On Lorentz transformations with elliptic biquaternions

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Abstract

In this study, the Lorentz transformations which are in accordant with special relativity have been examined for the first time with elliptic biquaternions. Since the elliptic biquaternions contain the complex structure, it is beneficial to examine the Lorentz transformations which can be the building blocks of relativistic physics via elliptic biquaternions. Therefore, as a result of relativistic transformation relation, it has been seen that the Lorentz transformations can be expressed with elliptic biquaternions and some special results have been given. In addition, matrix representations of obtained mathematical expressions are given. Thanks to the matrix representations of elliptical biquaternions, the property of commutativeness which is not valid for elliptic biquaternions has been eliminated and these representations provide a convenience for relativistic transformation relation. In this context, the presented method in this article is very useful.

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1 Introduction

Quaternions were discovered in 1843 during the studies of Irish mathematician William Rowan Hamilton while extending the complex numbers into three-dimensional space. From that day forward, the quaternions were used in physics by E. Schrödinger, W. Heisenberg, P. A. M. Dirac, M. Born and many other famous physicists between 1927 and 1932 in parallel with the developments of quantum mechanics. Real quaternions have applications in many fields such as differential geometry, motion geometry, quantum mechanics and real quaternions sentence can be represented as

$$\mathbb{H} = \{\mathbf{Q} = a_0 + a_1 i + a_2 j + a_3 k : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

where the quaternion unit bases $1, i, j$ and k satisfies the multiplication laws as follows:

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j$$

[1]. Also, the quaternionic units i, j, k are taken as base vectors of the $3-$ dimensional vector space, so that a quaternion $\mathbf{Q}$ can be written as a linear combination of a scalar and a spatial vector [2]. In 1853 W. R. Hamilton defined the sentence biquaternions (complex quaternions). The quaternionic units i, j, k are the same with the units in the real quaternions, this sentence is defined by

$$\mathbb{H}_{\mathbb{C}} = \{\mathbf{Q} = W + Xi + Yj + Zk : W, X, Y, Z \in \mathbb{C}\}$$

[3]. Complex quaternions have many applications in mathematics and physics. The applications of complex quaternions in physics have been mostly in the field of general and special relativity, relativistic mechanics, electromagnetism and quantum mechanics [4-9]. One of these is the expression

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of the particle mechanical equations, the conservation of 4-momentum formulas and the equations of electromagnetism by means of complex quaternions [10]. It is seen that many researchers have searched out the studies using the formulation of complex quaternions on Lorentz transformations and Maxwell equations [11-15]. Afterwards, it is seen that the complex quaternion algebra and the whole 2×2 complex matrix algebra are isomorphic to each other. This isomorphism is defined as

$$\psi : \mathbb{H}_{\mathbb{C}} \to M_2(\mathbb{C}) \ , \ \psi(a_0 + a_1 i + a_2 j + a_3 k) = \begin{bmatrix} a_0 + a_1 i & -a_2 - a_3 i \\ a_2 - a_3 i & a_0 - a_1 i \end{bmatrix}$$

[16-19]. By means of this isomorphism, many researchers have done studies. For instance, the matrix expressions of Lorentz transformations with complex quaternions studied by S. Demir [20]. On the other hand, the isomorphism for the 2x2 elliptical matrix representation of elliptical biquaternions is defined by Özen et al. [21] as follows:

$$\sigma : \mathbb{H}\mathbb{C}_p \to M_2(\mathbb{C}_p), \mathbf{Q} = A_0 + A_1 i + A_2 j + A_3 k \to \sigma\left(\mathbf{Q}\right) = \begin{bmatrix} A_0 + \frac{1}{\sqrt{|p|}} I A_1 & -A_2 - \frac{1}{\sqrt{|p|}} I A_3 \\ A_2 - \frac{1}{\sqrt{|p|}} I A_3 & A_0 - \frac{1}{\sqrt{|p|}} I A_1 \end{bmatrix} \quad (1.1)$$

where $\mathbf{Q}$ and $\mathbf{R}$ are any elliptic biquaternions, the function σ is a linear isomorphism bijection and surjection which provides properties $\sigma(\mathbf{Q}+\mathbf{R}) = \sigma(\mathbf{Q}) + \sigma(\mathbf{R})$ $\sigma(\mathbf{QR}) = \sigma(\mathbf{Q})\sigma(\mathbf{R})$. $\mathbb{H}\mathbb{C}_p$ and $M_2(\mathbb{C}_p)$ denote the set of elliptic biquaternions and the set of elliptic matrices of type 2×2 respectively. In this study, we investigate Lorentz transformations with elliptic biquaternions through the isomorphism described above. We define special matrices with elliptic biquaternions that we obtained under this isomorphism and using these special matrices we give the 4×4 elliptic and 8×8 type real matrices. Also, by means of matrix representations corresponding to the left Hamiltonian operator, the problem of the non-commutative property of the elliptic biquaternions in the algebraic structure has disappeared. Then we expressed the elliptic biquaternion $\mathbf{R}$ that relates space and time with equation 4.1. In Theorem 4.1, we give the elliptical matrix representation of this expression. In this way, the space and time components of the elliptic biquaternion $\mathbf{R}'$ obtained as a result of the relativistic transformation relation can be easily seen. In this expression, the real component of the elliptic biquaternion $\mathbf{R}' = ct + I\boldsymbol{r}$ denotes time and its the imaginer component denotes space. Note here that the imaginer component $I = \sqrt{|p|}$ denotes the space. Moreover, by taking $I^2 = p = -1$ since $I^2 = p < 0$, we also showed that elliptic biquaternions include complex cases.

2 Complex numbers and elliptic biquaternions

2.1 $p-$Complex numbers

The pairs (x, y) whose elements are real numbers are called complex numbers. The first persons to benefit from complex numbers are G. Cardan and R. Bombelli. On the other hand, the generalized complex numbers are separable to three as ordinary, dual and double complex numbers. The complex numbers defined as $i^2 = -1$ in case of imaginary unit i. It is the case the imaginary unit i the natural complex numbers are $i^2 = -1$. The English geometrician W. Clifford developed double complex numbers for the case of $i^2 = 1$. The German geometrician E. Study then carried out the studies on kinematics and line geometry, introducing different theorems and in the case of $i^2 = 0$ the dual numbers are obtained. I. Yaglom expressed that the ordinary, dual and double numbers are special members of the two-parameter family of the complex number system and he defined generalized complex numbers as

$$z = x + iy \ (x, y \in \mathbb{R}) \ , \ i^2 = iq + p \ (q, p \in \mathbb{R})$$

[22]. Harkin et al. have expressed that the generalized complex numbers are isomorphic to other number systems under consideration $i^2 = p$ and $q = 0$ in $i^2 = iq + p$. Then $p-$complex numbers system $\mathbb{C}_p$ is defined as

$$\mathbb{C}_p = \left\{ x + iy : \ x, y \in \mathbb{R}, \ i^2 = p \right\}.$$

Also, $p-$ trigonometric functions are defined as

$$\cos_p\theta_p = \begin{cases} \cos\ (\theta_p\sqrt{|p|}) & , p < 0 \\ 1 & , p = 0 \\ \cosh(\theta_p\sqrt{p}) & , p > 0 \end{cases} \tag{2.1}$$

and

$$sin_p\theta_p = \begin{cases} \frac{1}{\sqrt{|p|}} sin\ (\theta_p\sqrt{|p|}) & , p < 0 \\ 1 & , p = 0 \\ \frac{1}{\sqrt{p}} sin(\theta_p\sqrt{p}) & , p > 0 \end{cases} \tag{2.2}$$

[23]. Also Harkin expressed in his this geometric study that generalized complex numbers and rotations can be applied to the theory of special relativity in physics. In another studies on $p-$ complex numbers, the generalized Steiner formula and the Holditch theorem were presented by T. Erişir and M. A. Güngör [24]-[25]. The exponential function for e^{φ_p} an elliptic number $\varphi_p = x + Iy \in \mathbb{C}_p$ in the set of elliptic numbers $\mathbb{C}_p$ is known as a $p-$ analytic function defined in the form of $e^{\varphi_p} = e^{x+Iy} = e^x e^{Iy} = e^x(\cos_p y + I\sin_p y)$. From here, the $p-$ trigonometric functions can be obtained as follows:

$$\begin{cases} \cos_p y = \frac{e^{Iy}+e^{-Iy}}{2}, \ \sin_p y = \frac{e^{Iy}-e^{-Iy}}{2I} & \text{for} \quad x = 0 \\ \cos_p(Iy) = \frac{e^{py}+e^{-py}}{2} = \cosh(py) & \text{for} \quad y = Iy \\ \sin_p(Iy) = \frac{e^{py}-e^{-py}}{2I} = \frac{I}{p}\sinh(py) & \text{for} \quad y = Iy \end{cases} \tag{2.3}$$

On the other hand, for the elliptic complex variable $p-$ cosine and $p-$ sine functions also provide the identity ${\cos_p}^2\varphi_p + |p|\,{\sin_p}^2\varphi_p = 1$ where $\varphi_p = x + Iy \in \mathbb{C}_p$ [26].

2.2 Elliptic biquaternions

The set of elliptic biquaternions is given in the cartesian form as follows:

$$\mathbb{HC}_p = \left\{ \mathbf{Q} = \mathbf{q} + I\mathbf{q}' = A_0e_0 + A_1e_1 + A_2e_2 + A_3e_3 : \ A_0, A_1, A_2, A_3 \in \mathbb{C}_p, I^2 = p < 0 \right\}$$

where the numbers $A_i = q_i + Iq_i'$, $0 \leqslant i \leqslant 3$ state elliptic numbers. The unit bases of a biquaternion provide the product rule given in Table 1 [1]. Let $\mathbf{Q} = A_0e_0 + A_1e_1 + A_2e_2 + A_3e_3 \in \mathbb{HC}_p$ be an elliptic biquaternion providing the product rule in Table 1, then $\mathbf{Q}$ is defined as pure elliptic biquaternions provided that $A_0 = 0$ [27].

Any elliptic biquaternion $\mathbf{P}$ is expressed with the scalar (B_0) and vectorial components $(B_1e_1 + B_2e_2 + B_3e_3)$ in the following form

$$\mathbf{P} = S(\mathbf{P}) + V(\mathbf{P}) = B_0 + \boldsymbol{P} \tag{2.4}$$

Table 1. Multiplication scheme of the quaternionic units.

$\times$	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	$-e_0$	e_3	$-e_2$
e_2	e_2	$-e_3$	- e_0	e_1
e_3	e_3	e_2	-e_1	- e_0

The sum and quaternionic product of any two elliptic biquaternions $\mathbf{Q} = A_0e_0 + A_1e_1 + A_2e_2 + A_3e_3$ and $\mathbf{P} = B_0e_0 + B_1e_1 + B_2e_2 + B_3e_3$ are defined as

$$\mathbf{P} + \mathbf{Q} = (B_0e_0 + B_1e_1 + B_2e_2 + B_3e_3) + (A_0e_0 + A_1e_1 + A_2e_2 + A_3e_3)$$

$$\begin{aligned}
\mathbf{P} + \mathbf{Q} &= (B_0 + A_0) + (B_1 + A_1)e_1 + (B_2 + A_2)e_2 + (B_3 + A_3)e_3 \\
\mathbf{PQ} &= (B_0e_0 + B_1e_1 + B_2e_2 + B_3e_3)(A_0e_0 + A_1e_1 + A_2e_2 + A_3e_3) \\
\mathbf{PQ} &= (B_0A_0 - B_1A_1 - B_2A_2 - B_3A_3) + (B_1A_0 + B_0A_1 + B_3A_2 - B_2A_3)e_1 \\
&\quad +(B_2A_0 - B_3A_1 + B_0A_2 + B_1A_3)e_2 + (B_3A_0 + B_2A_1 - B_1A_2 + B_0A_3)e_3 \ .
\end{aligned}$$

In other words, the quaternionic product of two elliptic biquaternions can also be given, in the form of

$$\mathbf{PQ} = (B_0 + \boldsymbol{B})(A_0 + \boldsymbol{A}) = B_0A_0 + B_0\boldsymbol{A} + A_0\boldsymbol{B} - \langle \boldsymbol{B}, \boldsymbol{A}\rangle + \boldsymbol{B} \wedge \boldsymbol{A} \tag{2.5}$$

where $\langle , \rangle$ and $\wedge$ indicate inner product and vector product in three-dimensional space. The elliptic biquaternion $\mathbf{Q} = A_0e_0 + A_1e_1 + A_2e_2 + A_3e_3 = S(\mathbf{Q}) + V(\mathbf{Q}) = \mathbf{q} + I\mathbf{q}'$ has three different conjugates given as follows:

$$\begin{aligned}
\bar{\mathbf{Q}} &= A_0e_0 - A_1e_1 - A_2e_2 - A_3e_3 = S\left(\mathbf{Q}\right) - V\left(\mathbf{Q}\right)(\text{quaternionic conjugate}) \\
\mathbf{Q}^* &= (A_0)^*e_0 + (A_1)^*e_1 + (A_2)^*e_2 + (A_3)^*e_3 \\
&= (S(\mathbf{Q}))^* + (V(\mathbf{Q}))^* = \mathbf{q} - I\mathbf{q}'(\text{complex conjugate}) \\
\mathbf{Q}^\dagger &= \left(\bar{\mathbf{Q}}\right)^* = \overline{(\mathbf{Q}^*)} = (A_0)^*e_0 - (A_1)^*e_1 - (A_2)^*e_2 - (A_3)^*e_3 \\
&= \left(S\left(\mathbf{Q}\right)\right)^* - \left(V\left(\mathbf{Q}\right)\right)^* = \bar{\mathbf{q}} - I\bar{\mathbf{q}}' \text{ (total conjugate)}
\end{aligned}$$

where $\bar{\mathbf{q}}$ and $\bar{\mathbf{q}}'$ indicate the biquaternionic conjugates of real quaternions $\mathbf{q}$ and $\mathbf{q}'$ respectively. On the other hand, the inner product of these elliptic biquaternions is defined in the following way:

$$\langle \mathbf{P}, \mathbf{Q}\rangle_Q = \frac{1}{2}(\bar{\mathbf{P}}\mathbf{Q} + \bar{\mathbf{Q}}\mathbf{P}) = \frac{1}{2}(\mathbf{P}\bar{\mathbf{Q}} + \mathbf{Q}\bar{\mathbf{P}})$$

where the symbol $\langle\rangle_Q$ indicates a quaternionic inner product. Using this inner product semi-norm of $\mathbf{Q}$ is expressed as follows:

$$N(\mathbf{Q}) = \langle \mathbf{Q}, \mathbf{Q}\rangle_Q = \mathbf{Q}\bar{\mathbf{Q}} = \bar{\mathbf{Q}}\mathbf{Q} = {A_0}^2 + {A_1}^2 + {A_2}^2 + {A_3}^2$$

Also, since $A_i = q_i + Iq_i' \in \mathbb{C}_p$ $(0 \leqslant i \leqslant 3)$ are in the above equation, it is seen that $N(\mathbf{Q})$ can be equal to zero while $\mathbf{Q} \neq 0$. Accordingly, in the algebra of the elliptic biquaternions $\mathbb{HC}_p$, there are some elliptic biquaternions $\mathbf{Q}$ that satisfy the equality $\mathbf{Q}\bar{\mathbf{Q}}$ even if $\mathbf{Q} \neq 0$ and the algebra of the elliptic biquaternions $\mathbb{HC}_p$ contains zero divisors. Therefore, semi-norm is used instead of the norm in the elliptic biquaternion space $\mathbb{HC}_p$ in order to be appropriateness to the general literature [26]. Provided the semi-norm of the elliptic biquaternion $\mathbf{Q}$ is different from zero then it's inverse is given by $\mathbf{Q^{-1}} = \frac{\bar{\mathbf{Q}}}{N(\mathbf{Q})}$. Also, the module of this elliptic biquaternion $\mathbf{Q}$ is defined as

$$N(\mathbf{Q}) = \mathbf{Q}\bar{\mathbf{Q}} = \bar{\mathbf{Q}}\mathbf{Q} = |\mathbf{Q}|^2$$

and indicated by $|\mathbf{Q}|$. In the case of

$$N(\mathbf{Q}) = \langle \mathbf{Q}, \mathbf{Q} \rangle_Q = \mathbf{Q}\bar{\mathbf{Q}} = \bar{\mathbf{Q}}\mathbf{Q} = {A_0}^2 + {A_1}^2 + {A_2}^2 + {A_3}^2 = 1 \tag{2.6}$$

is satisfied, then the elliptic biquaternion $\mathbf{Q}$ is called unit elliptic biquaternions [26]. The set of all unit elliptic biquaternions is given by
$\eta = \{\mathbf{Q} \in \mathbb{HC}_p : N(\mathbf{Q}) = 1\}$ and the set of all pure elliptic biquaternions whose semi-norm is equal to the absolute value of p such as

$$\zeta = \left\{ w = A_1e_1 + A_2e_2 + A_3e_3 : N(w) = {A_1}^2 + {A_2}^2 + {A_3}^2 = |p|, w^* = -w, w^2 = p \right\}.$$

Thus, for any elliptic biquaternion $\mathbf{Q}$ whose modulus never vanishes, this elliptic biquaternion satisfies the following equalities [26]

$$R = \left(\sqrt{{A_0}^2 + {A_1}^2 + {A_2}^2 + {A_3}^2}\right)_{\mathbb{C}_p}, \ \cos_p\varphi_p = \frac{A_0}{R}, \ \sin_p\varphi_p = \frac{\left(\sqrt{{A_1}^2 + {A_2}^2 + {A_3}^2}\right)_{\mathbb{C}_p}}{\sqrt{|p|}R}. \tag{2.7}$$

Accordingly, the following theorem can be given.

Theorem 2.1. Let $w \in \zeta$ be a pure elliptic biquaternion and $\varphi_p \in \mathbb{C}_p$ be an elliptical complex angle. In this case, the elliptic biquaternion can be written as $\cos_p\varphi_p + w\sin_p\varphi_p$ is a unit elliptic biquaternion [26].

Theorem 2.2. Let $\mathbf{Q} = A_0 + A_1e_1 + A_2e_2 + A_3e_3$ be the elliptic biquaternion whose modulus is non-zero. In this case, a pure elliptic biquaternion is derived from the vectorial part of such that

$$w_{\mathbf{Q}} = \sqrt{|p|}\frac{A_1e_1 + A_2e_2 + A_3e_3}{\left(\sqrt{{A_1}^2 + {A_2}^2 + {A_3}^2}\right)_{\mathbb{C}_p}} \tag{2.8}$$

is an element of the set ζ, i.e., $(w_{\mathbf{Q}})^2 = p$ [26].

The elliptic biquaternion $\mathbf{Q} = A_0 + A_1i + A_2j + A_3k$ in $p-$ trigonometric form depending on Theorem (2.1) and Theorem (2.2) can be written as

$$\mathbf{Q} = R(cos_p\varphi_p + w_{\mathbf{Q}}\sin_p\varphi_p). \tag{2.9}$$

Here the angle $\varphi_p \in \mathbb{C}_p$ is an elliptic complex angle in the form of $\varphi_p = x + Iy$ [26].

Theorem 2.3. If an elliptic biquaternion is given as $\mathbf{Q} = \cosh(p\frac{\theta_p}{2}) + \frac{1}{I}\hat{q}\sinh(p\frac{\theta_p}{2})$ then $\mathbf{Q}$ is a unit elliptic biquaternion

Proof. For a given velocity vector $\vec{v} = v_1e_1 + v_2e_2 + v_3e_3$ and pure unit elliptic biquaternion $\hat{q} = \sqrt{|p|}\frac{v_1e_1+v_2e_2+v_3e_3}{\sqrt{<\vec{v},\vec{v}>}}$, $\mathbf{Q} = \cosh(py) + \frac{1}{I}\hat{q}\sinh(py)$ can be written as unit elliptic biquaternion. By means of the equalities 2.3, 2.8 and 2.9 the conjugate of the elliptic biquaternion is taken as $\bar{\mathbf{Q}} = \cosh(py) - \frac{1}{I}\hat{q}\sinh(py)$. Here, if we give $y = \frac{\theta_p}{2}$, we can obtain as

$$\mathbf{Q}\bar{\mathbf{Q}} = N(\mathbf{Q}) = \cosh^2(p\frac{\theta_p}{2}) - \sinh^2(p\frac{\theta_p}{2}) = 1.$$

Q.E.D.

3 Matrix representations of elliptic biquaternions

In terms of more descriptive mathematical expressions, it is also the preferred method to give this mathematical expressions matrix representations. In this section, we study with matrices. It is possible to represent an elliptic biquaternion $\mathbf{Q}$ with 4×4 matrices. These matrices that similar to Pauli spin matrices can be defined with the help of isomorphism expressed in 1.1 as follows:

$$\begin{gathered}\sigma(e_0) = \sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \sigma(e_1) = \sigma_1 = \begin{bmatrix} \frac{I}{\sqrt{|p|}} & 0 \\ 0 & -\frac{I}{\sqrt{|p|}} \end{bmatrix} \\ \sigma(e_2) = \sigma_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \sigma(e_3) = \sigma_3 = \begin{bmatrix} 0 & -\frac{I}{\sqrt{|p|}} \\ -\frac{I}{\sqrt{|p|}} & 0 \end{bmatrix}.\end{gathered} \tag{3.1}$$

These matrices satisfy the properties of

$$(\sigma(e_0))^2 = 1,\ (\sigma(e_1))^2 = (\sigma(e_2))^2 = (\sigma(e_3))^2 = -I_2.$$

In addition with the help of these special matrices, we define the matrices corresponding to the base elements e_0, e_1, e_2, e_3 in the real quaternions as follows:

$$\begin{gathered}\Gamma_0 = \begin{bmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{bmatrix}_{2\times 2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \Gamma_1 = \begin{bmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \\ \Gamma_2 = \begin{bmatrix} 0 & -\sigma_0 \\ \sigma_0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \Gamma_3 = \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.\end{gathered} \tag{3.2}$$

These matrices satisfy the multiplication relations for basis of elliptic biquaternion as

$${\Gamma_0}^2 = \Gamma_0 = I_4, \quad {\Gamma_1}^2 = {\Gamma_2}^2 = {\Gamma_3}^2 = -I_4, \quad \Gamma_j\Gamma_k = \delta_{jk}\Gamma_0 - \varepsilon_{jkl}\Gamma_l \tag{3.3}$$

where the expressions δ and ε denote Kronecker Delta and Levi-Civita symbols, respectively. Thus for an elliptic biquaternion $\mathbf{Q} \cong A_0\Gamma_0 + A_1\Gamma_1 + A_2\Gamma_2 + A_3\Gamma_3$ by using quaternionic bases in 3.2

the known left Hamiltonian matrix can be given by

$$\mathbb{H}^-(\mathbf{Q}) = \begin{bmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & A_3 & -A_2 \\ A_2 & -A_3 & A_0 & A_1 \\ A_3 & A_2 & -A_1 & A_0 \end{bmatrix} \tag{3.4}$$

On the other hand, the known right Hamiltonian matrix is as follows:

$$\mathbb{H}^+(\mathbf{Q}) = \begin{bmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & +A_2 \\ A_2 & +A_3 & A_0 & -A_1 \\ A_3 & -A_2 & +A_1 & A_0 \end{bmatrix}.$$

The number of elliptic biquaternions $\mathbf{Q}$ here is also; it is also possible to represent with the matrix 4×1 given by $\mathbf{Q} = [A_0, \boldsymbol{A}]^T = \begin{bmatrix} A_0 & A_1 & A_2 & A_3 \end{bmatrix}^T$.
The following theorem can be given for Hamiltonian matrices.

Theorem 3.1. Let two elliptic biquaternions be $\mathbf{Q} = A_0\Gamma_0 + A_1\Gamma_1 + A_2\Gamma_2 + A_3\Gamma_3$ and $\mathbf{P} = B_0\Gamma_0 + B_1\Gamma_1 + B_2\Gamma_2 + B_3\Gamma_3$ in algebra $\mathbb{HC}_p$. Accordingly, the following properties are provided:

$\mathbf{Q} = \mathbf{R} \Leftrightarrow \mathbb{H}^+(\mathbf{Q}) = \mathbb{H}^+(\mathbf{R}) \Leftrightarrow \mathbb{H}^-(\mathbf{Q}) = \mathbb{H}^-(\mathbf{R})$

1. $\mathbb{H}^+(\mathbf{Q}+\mathbf{R}) = \mathbb{H}^+(\mathbf{Q}) + \mathbb{H}^+(\mathbf{R})$, $\quad \mathbb{H}^-(\mathbf{Q}+\mathbf{R}) = \mathbb{H}^-(\mathbf{Q}) + \mathbb{H}^-(\mathbf{R})$
2. $\mathbb{H}^+(\lambda\mathbf{Q}) = \lambda\mathbb{H}^+(\mathbf{Q})$, $\quad \mathbb{H}^-(\lambda\mathbf{Q}) = \lambda\mathbb{H}^-(\mathbf{Q})$
3. $\mathbb{H}^+(\mathbf{QR}) = \mathbb{H}^+(\mathbf{Q})\mathbb{H}^+(\mathbf{R})$, $\quad \mathbb{H}^-(\mathbf{QR}) = \mathbb{H}^-(\mathbf{R})\mathbb{H}^-(\mathbf{Q})$
4. $\mathbb{H}^+(\bar{\mathbf{Q}}) = [\mathbb{H}^+(\mathbf{Q})]^T$, $\quad \mathbb{H}^-(\bar{\mathbf{Q}}) = [\mathbb{H}^-(\mathbf{Q})]^T$
5. $\mathbb{H}^+(\mathbf{Q}^*) = \overline{\mathbb{H}^+(\mathbf{Q})}$, $\quad \mathbb{H}^-(\mathbf{Q}^*) = \overline{\mathbb{H}^-(\mathbf{Q})}$
6. $\mathbb{H}^+(\mathbf{Q}^\dagger) = [\mathbb{H}^+(\mathbf{Q})]^*$, $\quad \mathbb{H}^-(\mathbf{Q}^\dagger) = [\mathbb{H}^-(\mathbf{Q})]^*$

where is an elliptical number [27].

On the other hand, by using the above properties it is possible to define the elliptic biquaternion multiplication given in equation 2.5 as follows:

$$\mathbb{H}^-(\mathbf{P})\mathbf{Q} = \begin{bmatrix} B_0 & -B_1 & -B_2 & -B_3 \\ B_1 & B_0 & B_3 & -B_2 \\ B_2 & -B_3 & B_0 & B_1 \\ B_3 & B_2 & -B_1 & B_0 \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix}. \tag{3.5}$$

Now, let we the following antisymmetric matrices are define follow as:

$$\tilde{\boldsymbol{P}} = \begin{bmatrix} 0 & B_3 & -B_2 \\ -B_3 & 0 & B_1 \\ B_2 & -B_1 & 0 \end{bmatrix} \tag{3.6}$$

and

$$\tilde{\boldsymbol{Q}} = \begin{bmatrix} 0 & A_3 & -A_2 \\ -A_3 & 0 & A_1 \\ A_2 & -A_1 & 0 \end{bmatrix}. \tag{3.7}$$

In this case with the help of the defined matrix in 3.6 the expression 3.5 can be given as follows:

$$\mathbf{PQ} \cong \begin{bmatrix} B_0 & -\boldsymbol{B} \\ \boldsymbol{B^T} & B_0 I_3 + \tilde{\boldsymbol{P}} \end{bmatrix} \begin{bmatrix} A_0 \\ \boldsymbol{A} \end{bmatrix}. \tag{3.8}$$

Similarly using the right Hamiltonian matrix representation of multiplication the equality can be written as follows:

$$\mathbb{H}^{+}(\mathbf{Q})\mathbf{P} = \begin{bmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & A_2 \\ A_2 & A_3 & A_0 & -A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix}. \tag{3.9}$$

On the other hand, considering the elliptic biquaternionic matrix given in 3.7, this multiplication matrix can be expressed as an elliptic matrix of size 2x2

$$\mathbf{PQ} = \begin{bmatrix} A_0 & -\boldsymbol{B} \\ \boldsymbol{B^T} & A_0 I_3 - \tilde{\boldsymbol{Q}} \end{bmatrix} \begin{bmatrix} B_0 \\ \boldsymbol{B} \end{bmatrix} \tag{3.10}$$

As is known, the product of two elliptic biquaternions has no commutative property. Therefore we define two elliptic biquaternions as follows:

$$\breve{\boldsymbol{P}} = \begin{bmatrix} B_0 & -\boldsymbol{B} \\ \boldsymbol{B^T} & B_0 I_3 + \tilde{\boldsymbol{P}} \end{bmatrix} \tag{3.11}$$

and

$$\breve{\mathbf{Q}} = \begin{bmatrix} A_0 & -\boldsymbol{A} \\ \boldsymbol{A^T} & A_0 I_3 - \tilde{\boldsymbol{Q}} \end{bmatrix} \tag{3.12}$$

$$\mathbb{H}^{-}(\mathbf{P})\mathbf{Q} \cong \breve{\mathbf{P}}\mathbf{Q}$$

and

$$\mathbb{H}^{+}(\mathbf{Q})\mathbf{P} \cong \breve{\mathbf{Q}}\mathbf{P}.$$

Hence, we can write the following equality

$$\mathbb{H}^{-}(\mathbf{P})\mathbf{Q} = \mathbb{H}^{+}(\mathbf{Q})\mathbf{P}$$

and

$$\breve{\mathbf{P}}\mathbf{Q} = \breve{\mathbf{Q}}\mathbf{P}. \tag{3.13}$$

Thus, the commutative property which is not valid for elliptic biquaternions can be easily obtained with the help of matrices. This equality will provide convenience for matrix representations of the relativistic transformation relation. It is also possible to express the elliptic biquaternion $\mathbf{Q} = \mathbf{q} + I\mathbf{q}'$ consisting of eight real components with the 8×8 real matrix using the expressions 3.2 defined for the base elements of the real quaternions.

Theorem 3.2. For the elliptical biquaternion $\mathbf{Q}$ there is the relation:
$\mathbb{H}^-(\mathbf{Q}) \cong (a_0 + Ta'_0)\xi_0 + (a_1 + Ta'_1)\xi_1 + (a_2 + Ta'_2)\xi_2 + (a_3 + Ta'_3)\xi_3$
$\mathbb{H}^-(\mathbf{Q}) \cong$

$$\left[\begin{array}{cccccccc} a_0 & -a_1 & -a_2 & -a_3 & \sqrt{|p|}a_0{}' & -\sqrt{|p|}a_1{}' & -\sqrt{|p|}a_2{}' & -\sqrt{|p|}a_3{}' \\ a_1 & a_0 & a_3 & -a_2 & \sqrt{|p|}a_1{}' & \sqrt{|p|}a_0{}' & \sqrt{|p|}a_3{}' & -\sqrt{|p|}a_2{}' \\ a_2 & -a_3 & a_0 & a_1 & \sqrt{|p|}a_2{}' & -\sqrt{|p|}a_3{}' & \sqrt{|p|}a_0{}' & \sqrt{|p|}a_1{}' \\ a_3 & a_2 & -a_1 & a_0 & \sqrt{|p|}a_3{}' & \sqrt{|p|}a_2{}' & -\sqrt{|p|}a_1{}' & \sqrt{|p|}a_0{}' \\ -\sqrt{|p|}a_0{}' & \sqrt{|p|}a_1{}' & \sqrt{|p|}a_2{}' & \sqrt{|p|}a_3{}' & a_0 & -a_1 & -a_2 & -a_3 \\ -\sqrt{|p|}a_1{}' & -\sqrt{|p|}a_0{}' & -\sqrt{|p|}a_3{}' & \sqrt{|p|}a_2{}' & a_1 & a_0 & a_3 & -a_2 \\ -\sqrt{|p|}a_2{}' & \sqrt{|p|}a_3{}' & -\sqrt{|p|}a_0{}' & -\sqrt{|p|}a_1{}' & a_2 & -a_3 & a_0 & a_1 \\ -\sqrt{|p|}a_3{}' & -\sqrt{|p|}a_2{}' & \sqrt{|p|}a_1{}' & -\sqrt{|p|}a_0{}' & a_3 & a_2 & -a_1 & a_0 \end{array}\right]$$

$$\cong \left[\begin{array}{cc} \mathbb{H}^-(\mathbf{Q}) & \sqrt{|p|}\mathbb{H}^-(\mathbf{Q}') \\ -\sqrt{|p|}\mathbb{H}^-(\mathbf{Q}') & \mathbb{H}^-(\mathbf{Q}) \end{array}\right]$$

Proof. For the elliptic biquaternion $\mathbf{Q}$ the matrix representation is

$$\mathbb{H}^-(\mathbf{Q}) \cong (a_0 + Ta'_0)\xi_0 + (a_1 + Ta'_0)\xi_1 + (a_2 + Ta'_0)\xi_2 + (a_3 + Ta'_0)\xi_3 \tag{3.14}$$

First, let us define the following matrix

$$T = \mu \times \Gamma_0 = \left[\begin{array}{cc} 0 & \Gamma_0\sqrt{|p|} \\ -\Gamma_0\sqrt{|p|} & 0 \end{array}\right]$$

$$T = \left[\begin{array}{cccccccc} 0 & 0 & 0 & 0 & \sqrt{|p|} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{|p|} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{|p|} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{|p|} \\ -\sqrt{|p|} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{|p|} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{|p|} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{|p|} & 0 & 0 & 0 & 0 \end{array}\right] \tag{3.15}$$

where the quaternionic units provide the multiplication rules

$$\xi_0{}^2 = -\xi_j{}^2 = I_8 = \xi_0,\ \xi_1\xi_2 = -\xi_3,\ \xi_2\xi_3 = -\xi_1,\ \xi_3\xi_1 = -\xi_2\ \ \xi_2\xi_1 = \xi_3\ ,\ \xi_3\xi_2 = \xi_1\ ,\ \xi_1\xi_3 = \xi_2. \tag{3.16}$$

Also, let us define matrices of ξ_j, $j = 0, 1, 2, 3$ and elliptic matrix μ, 2×2 as

$$\xi_j = \sigma_0 \times \Gamma_j = \left[\begin{array}{cc} \Gamma_j & 0 \\ 0 & \Gamma_j \end{array}\right]$$

and

$$\mu = \left[\begin{array}{cc} 0 & \sqrt{|p|} \\ -\sqrt{|p|} & 0 \end{array}\right].$$

Let us find the 8×8 real matrix representation of the equation given in Theorem 3.2. Firstly,

$$a_0\xi_0 = \begin{bmatrix} a_0\Gamma_0 & 0 \\ 0 & a_0\Gamma_0 \end{bmatrix} = \begin{bmatrix} a_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_0 \end{bmatrix}$$

and

$$(Ta_0{}')\xi_0 = \begin{bmatrix} 0 & {\Gamma_0}^2\sqrt{|p|}a_0{}' \\ -{\Gamma_0}^2\sqrt{|p|}a_0{}' & 0 \end{bmatrix}.$$

So, we can clearly write that

$$(Ta_0{}')\xi_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & a_0{}'\sqrt{|p|} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_0{}'\sqrt{|p|} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_0{}'\sqrt{|p|} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_0{}'\sqrt{|p|} \\ -a_0{}'\sqrt{|p|} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a_0{}'\sqrt{|p|} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_0{}'\sqrt{|p|} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_0{}'\sqrt{|p|} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, we obtain the matrix sum of the two equations as follows:

$$a_0\xi_0 + (Ta_0{}')\xi_0 = \begin{bmatrix} q_0 & 0 & 0 & 0 & \sqrt{|p|}a_0{}' & 0 & 0 & 0 \\ 0 & q_0 & 0 & 0 & 0 & \sqrt{|p|}a_0{}' & 0 & 0 \\ 0 & 0 & q_0 & 0 & 0 & 0 & \sqrt{|p|}a_0{}' & 0 \\ 0 & 0 & 0 & q_0 & 0 & 0 & 0 & \sqrt{|p|}a_0{}' \\ -\sqrt{|p|}a_0{}' & 0 & 0 & 0 & q_0 & 0 & 0 & 0 \\ 0 & -\sqrt{|p|}a_0{}' & 0 & 0 & 0 & q_0 & 0 & 0 \\ 0 & 0 & -\sqrt{|p|}a_0{}' & 0 & 0 & 0 & q_0 & 0 \\ 0 & 0 & 0 & -\sqrt{|p|}a_0{}' & 0 & 0 & 0 & q_0 \end{bmatrix}.$$

Similarly, their matrices of addition can be calculated according to other unit bases of the elliptic biquaternion. Hence, we obtain the matrix representation in 8×8 dimension of the elliptic biquaternion $\mathbf{Q}$ as follows:

$$\mathbb{H}^-(\mathbf{Q}) \cong \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 & \sqrt{|p|}a'_0 & -\sqrt{|p|}a'_1 & -\sqrt{|p|}a'_2 & -\sqrt{|p|}a'_3 \\ a_1 & a_0 & a_3 & -a_2 & \sqrt{|p|}a'_1 & \sqrt{|p|}a'_0 & \sqrt{|p|}a'_3 & -\sqrt{|p|}a'_2 \\ a_2 & -a_3 & a_0 & a_1 & \sqrt{|p|}a'_2 & -\sqrt{|p|}a'_3 & \sqrt{|p|}a'_0 & \sqrt{|p|}a'_1 \\ a_3 & a_2 & -a_1 & a_0 & \sqrt{|p|}a'_3 & \sqrt{|p|}a'_2 & -\sqrt{|p|}a'_1 & \sqrt{|p|}a'_0 \\ -\sqrt{|p|}a'_0 & \sqrt{|p|}a'_1 & \sqrt{|p|}a'_2 & \sqrt{|p|}a'_3 & a_0 & -a_1 & -a_2 & -a_3 \\ -\sqrt{|p|}a'_1 & -\sqrt{|p|}a'_0 & -\sqrt{|p|}a'_3 & \sqrt{|p|}a'_2 & a_1 & a_0 & a_3 & -a_2 \\ -\sqrt{|p|}a'_2 & \sqrt{|p|}a'_3 & -\sqrt{|p|}a'_0 & -\sqrt{|p|}a'_1 & a_2 & -a_3 & a_0 & a_1 \\ -\sqrt{|p|}a'_3 & -\sqrt{|p|}a'_2 & \sqrt{|p|}a'_1 & -\sqrt{|p|}a'_0 & a_3 & a_2 & -a_1 & a_0 \end{bmatrix}$$

and thus, we get

$$\mathbb{H}^{-}(\mathbf{Q}) \cong \begin{bmatrix} \mathbb{H}^{-}(\mathbf{Q}) & \sqrt{|p|}\mathbb{H}^{-}(\mathbf{Q}') \\ -\sqrt{|p|}\mathbb{H}^{-}(\mathbf{Q}') & \mathbb{H}^{-}(\mathbf{Q}) \end{bmatrix}.$$

Q.E.D.

4 Lorentz transformations with elliptic biquaternions

Lorentz transformation in physics is named after Dutch physicist H. Lorentz. The transformations explain how the speed of light independent of the frame of reference is to be observed the measurements of space and time measured by the two observers being related. Lorentz transformations are linear transformations in concordance with special relativity. Every elliptic biquaternion $\mathbf{R}$ that combines the cartesian position vector $\vec{r} = r_1\hat{i} + r_2\hat{j} + r_3\hat{k}$ with the time t can be written as follows:

$$\mathbf{R} = ct + I\boldsymbol{r} = cte_0 + I(r_1e_1 + r_2e_2 + r_3e_3) = R_0e_0 + R_1e_1 + R_2e_2 + R_3e_3 \tag{4.1}$$

where c is the speed of light in four-dimensional space, $\boldsymbol{r}$ is the pure real quaternion. Considering the left Hamiltonian matrix and Theorem 3.2 for the elliptic biquaternion $\mathbf{R} = ct + I\boldsymbol{r}$ the following theorem can be given.

Theorem 4.1. Let us consider an elliptical biquaternion $\mathbf{R} = ct + I\mathbf{r}$ that combines vector of position and time in four-dimensional space. The elliptic biquaternionic representation of $\mathbb{H}^{-}(\mathbf{R})$ is as follows

$$\mathbb{H}^{-}(\mathbf{R}) \cong \begin{bmatrix} ct\Gamma_0 & \sqrt{|p|}\mathbb{H}^{-}(\mathbf{R}) \\ -\sqrt{|p|}\mathbb{H}^{-}(\mathbf{R}) & ct\Gamma_0 \end{bmatrix}$$

where c indicate the speed of light.

Proof. Matrices corresponding to the base elements of the pure elliptic biquaternion $\boldsymbol{R}$ are written as follows

$$\Gamma_1 = \begin{bmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} 0 & -\sigma_0 \\ \sigma_0 & 0 \end{bmatrix}, \qquad \Gamma_3 = \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix}.$$

Considering these matrices the equality, we get

$$\mathbb{H}^{-}(\mathbf{R}) = r_1\Gamma_1 + r_2\Gamma_2 + r_3\Gamma_3$$

$$\mathbb{H}^{-}(\mathbf{R}) = \begin{bmatrix} 0 & -r_1 & 0 & 0 \\ r_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_1 \\ 0 & 0 & -r_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -r_2 & 0 \\ 0 & 0 & 0 & -r_2 \\ r_2 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -r_3 \\ 0 & 0 & r_3 & 0 \\ 0 & -r_3 & 0 & 0 \\ r_3 & 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -r_1 & -r_2 & -r_3 \\ r_1 & 0 & r_3 & -r_2 \\ r_2 & -r_3 & 0 & r_1 \\ r_3 & r_2 & -r_1 & 0 \end{bmatrix}.$$

Also, from the equalities 3.15 and 3.16, we obtain the real matrix in dimension 8×8 as follows

$$\mathbb{H}^{-}(\mathbf{R}) = ct\xi_0 + (Tr_1)\xi_1 + (Tr_2)\xi_2 + (Tr_3)\xi_3 \tag{4.2}$$

$$\mathbb{H}^{-}(\mathbf{R}) \cong \begin{bmatrix} ct & 0 & 0 & 0 & 0 & -r_1\sqrt{|p|} & -r_2\sqrt{|p|} & -r_3\sqrt{|p|} \\ 0 & ct & 0 & 0 & r_1\sqrt{|p|} & 0 & r_3\sqrt{|p|} & -r_2\sqrt{|p|} \\ 0 & 0 & ct & 0 & r_2\sqrt{|p|} & -r_3\sqrt{|p|} & 0 & r_1\sqrt{|p|} \\ 0 & 0 & 0 & ct & r_3\sqrt{|p|} & r_2\sqrt{|p|} & -r_1\sqrt{|p|} & 0 \\ 0 & r_1\sqrt{|p|} & r_2\sqrt{|p|} & r_3\sqrt{|p|} & ct & 0 & 0 & r_3\sqrt{|p|} \\ -r_1\sqrt{|p|} & 0 & -r_3\sqrt{|p|} & r_2\sqrt{|p|} & 0 & ct & -r_3\sqrt{|p|} & 0 \\ -r_2\sqrt{|p|} & r_3\sqrt{|p|} & 0 & -r_1\sqrt{|p|} & 0 & r_3\sqrt{|p|} & ct & 0 \\ -r_3\sqrt{|p|} & -r_2\sqrt{|p|} & r_1\sqrt{|p|} & 0 & -r_3\sqrt{|p|} & 0 & 0 & ct \end{bmatrix}.$$

Hence, 2×2 elliptical matrix representation of the resulting matrix is as follows:

$$\mathbb{H}^{-}(\mathbf{R}) \cong \begin{bmatrix} ct\Gamma_0 & \sqrt{|p|}\mathbb{H}^{-}(\mathbf{R}) \\ -\sqrt{|p|}\mathbb{H}^{-}(\mathbf{R}) & ct\Gamma_0 \end{bmatrix}.$$

Q.E.D.

As a result of the relativistic transformation relation of the elliptic biquaternion $\mathbf{R} = ct + I\boldsymbol{r}$ Lorentz transformation equations can be obtained.
Firstly, let us the unit elliptic biquaternion $\mathbf{Q}$ which can be define as

$$\mathbf{Q} = a_0 + I\boldsymbol{a'} = A_0e_0 + A_1e_1 + A_2e_2 + A_3e_3 = A_0 + \boldsymbol{A}. \tag{4.3}$$

Considering Theorem 2.3 we state the unit elliptic biquaternion as follows:

$$\mathbf{Q} = \cosh(p\frac{\theta_p}{2}) + \frac{1}{I}\hat{q}\sinh(p\frac{\theta_p}{2}) = \cosh(p\frac{\theta_p}{2}) + \frac{1}{I}I\frac{\vec{v}}{\sqrt{<\vec{v},\vec{v}>}}\sinh(p\frac{\theta_p}{2}). \tag{4.4}$$

Here, we consider a speed $\vec{v} = v_1e_1 + v_2e_2 + v_3e_3$ which is in the opposite direction. Thus, from 4.3 and 4.4 the following equation holds:

$$a_0 = A_0 = \cosh(p\frac{\theta_p}{2})\ ,\ \boldsymbol{a'} = \frac{1}{I}\frac{\vec{v}}{\nu}\sinh(p\frac{\theta_p}{2}) = \frac{\hat{q}}{p}\sinh(p\frac{\theta_p}{2}),\ \boldsymbol{A} = \frac{\vec{v}}{\nu}\sinh(p\frac{\theta_p}{2}). \tag{4.5}$$

On the other hand from the known Lorentz coordinate transformations, we get

$$\begin{aligned} \cosh(p\theta_p) &= \frac{|c|}{\sqrt{c^2-\nu^2}} = \frac{|c|}{|c|\sqrt{1-\frac{\nu^2}{c^2}}} = \frac{1}{1-|p|\,\nu^2} = \beta \\ \sinh(p\theta_p) &= \frac{|\nu|}{\sqrt{c^2-\nu^2}} = \frac{|\nu|}{|c|\sqrt{1-|p|\,v^2}} = \mp\frac{\nu}{c\sqrt{1-p\nu^2}} = \mp\frac{\beta}{c}\nu \end{aligned} \tag{4.6}$$

where $|p| = \frac{1}{c^2}$ and $\sqrt{<\vec{v},\vec{v}>} = \nu$. Also, from the 4.1 and we obtain

$$\begin{aligned} a_0 = A_0, \quad A = Ia', \quad A_1 = Ia'_1, \quad A_2 = Ia'_2, \\ A_3 = Ia'_3, \quad a'_0e_0 + \quad a'_1e_1 + a'_2e_2 + a'_3e_3 = a', R_0 = ct, \quad \boldsymbol{R} = I\boldsymbol{r}. \end{aligned} \tag{4.7}$$

The elliptic biquaternion obtained from 4.1 as a result of the relation of the relativistic transformation of elliptic biquaternion $\mathbf{R} = ct + I\boldsymbol{r}$ can be written as $\mathbf{R}' = ct + I\boldsymbol{r'}$. The relativistic transformation relation of elliptic biquaternion $\mathbf{R}$ is expressed as follows

$$\mathbf{R}' = \mathbf{Q}\mathbf{R}\bar{\mathbf{Q}}^*. \tag{4.8}$$

This equality is rewritten using equations 3.15 and Theorem 4.1 as

$$\begin{aligned}\mathbb{H}^-(\mathbf{R}') &= \mathbb{H}^-(\mathbf{Q})\mathbb{H}^-(\mathbf{R})\mathbb{H}^-(\bar{\mathbf{Q}})^* \\ &= (a_0\xi_0 + (Ta_1)\xi_1 + (Ta_2)\xi_2 + (Ta_3)\xi_3)(t\xi_0 + (Tr_1)\xi_1 + (Tr_2)\xi_2 \\ &\quad +(Tr_3)\xi_3)(a_0\xi_0 + (Ta_1)\xi_1 + (Ta_2)\xi_2 + (Ta_3)\xi_3)\end{aligned} \tag{4.9}$$

or in the other way

$$\mathbb{H}^-(\mathbf{R}') \cong \begin{bmatrix} a_0\Gamma_0 & -\sqrt{|p|}\mathbb{H}^-(\mathbf{Q}) \\ -\sqrt{|p|}\mathbb{H}^-(\mathbf{Q}) & a_0\Gamma_0 \end{bmatrix} \begin{bmatrix} ct\Gamma_0 & \sqrt{|p|}\mathbb{H}^-(\boldsymbol{R}) \\ -\sqrt{|p|}\mathbb{H}^-(\boldsymbol{R}) & ct\Gamma_0 \end{bmatrix} \begin{bmatrix} a_0\Gamma_0 & \sqrt{|p|}\mathbb{H}^-(\mathbf{Q}) \\ -\sqrt{|p|}\mathbb{H}^-(\mathbf{Q}) & a_0\Gamma_0 \end{bmatrix} \tag{4.10}$$

is represented. Also by equality 4.8 we write

$$\begin{aligned}\mathbf{R}' &= (A_0 + \boldsymbol{A})(A_0R_0 + R_0\boldsymbol{A} + A_0\boldsymbol{R} - \langle \boldsymbol{R}, \boldsymbol{A}\rangle + (\boldsymbol{R} \wedge A) \\ &= {A_0}^2R_0 + A_0R_0\boldsymbol{A} + {A_0}^2\boldsymbol{R} - A_0 < \boldsymbol{R}, \boldsymbol{A} > +A_0 < \boldsymbol{R}, \boldsymbol{A} > +A_0R_0\boldsymbol{A} \\ &\quad - R_0 < \boldsymbol{A}, \boldsymbol{A} > -A_0 < \boldsymbol{A}, \boldsymbol{R} > \\ &= A_0\,\langle \boldsymbol{A}, \boldsymbol{R}\rangle - \boldsymbol{A}\,\langle \boldsymbol{R}, \boldsymbol{A}\rangle - \langle \boldsymbol{A}, \boldsymbol{R} \wedge \boldsymbol{A}\rangle + \boldsymbol{A} \wedge (\boldsymbol{R} \wedge \boldsymbol{A}).\end{aligned}$$

Since there is $\langle \boldsymbol{A}, \boldsymbol{R} \wedge \boldsymbol{A}\rangle = 0$ we reach the following general equality for $\mathbf{R}'$:

$$\begin{aligned}\mathbf{R}' = {A_0}^2R_0 + 2A_0R_0A + {A_0}^2\boldsymbol{R} - A_0 < \boldsymbol{R}, \boldsymbol{A} > -R_0 < \boldsymbol{A}, \boldsymbol{A} > \\ -A_0 < \boldsymbol{A}, \boldsymbol{R} > -\boldsymbol{A} < \boldsymbol{R}, \boldsymbol{A} > +\boldsymbol{A} \wedge (\boldsymbol{R} \wedge \boldsymbol{A}).\end{aligned} \tag{4.11}$$

From the properties of the vector product, we get

$$\boldsymbol{A} \wedge (\boldsymbol{R} \wedge \boldsymbol{A}) = \boldsymbol{R} < \boldsymbol{A}, \boldsymbol{A} > -\boldsymbol{A} < \boldsymbol{A}, \boldsymbol{R} >= \boldsymbol{A}^{\mathbf{2}}\boldsymbol{R} - \boldsymbol{A} < \boldsymbol{A}, \boldsymbol{R} > .$$

If this equality is written in the equation 4.11, we obtain

$$\mathbf{R}' = {A_0}^2R_0 + 2A_0R_0\boldsymbol{A} + {A_0}^2\boldsymbol{R} - 2A_0 < \boldsymbol{A}, \boldsymbol{R} > -R_0\boldsymbol{A}^{\mathbf{2}} - 2\boldsymbol{A} < \boldsymbol{A}, \boldsymbol{R} > +\boldsymbol{A}^{\mathbf{2}}\boldsymbol{R}.$$

By substituting 4.7 in this equation, we reach the following equation

$$\begin{aligned}\mathbf{R}' &= {A_0}^2ct - ct(I\boldsymbol{a'})^2 - 2A_0I^2 < \boldsymbol{r}, \boldsymbol{a'} > +{A_0}^2(I\boldsymbol{r}) + 2A_0ct(I\boldsymbol{a'}) \\ &\quad - 2I^3\boldsymbol{a'} < \boldsymbol{a'}, \boldsymbol{r} > +(I\boldsymbol{a'})^2(I\boldsymbol{r}) \\ &= ({A_0}^2 - p(\boldsymbol{a'})^2)ct - 2A_0p < \boldsymbol{r}, \boldsymbol{a'} > +I\boldsymbol{r}({A_0}^2 + p(\boldsymbol{a'})^2) + 2IctA_0\boldsymbol{a'} \\ &\quad - 2Ip\boldsymbol{a'} < \boldsymbol{a'}, \boldsymbol{r} >.\end{aligned}$$

In this expression, the terms containing and not containing I ($I^2 = p < 0$) must be matched so that both sides of equality can be equal to each other. Thus, for relativistic transformation relation of space and time as the equation $\mathbf{R}' = ct' + I\mathbf{r}'$ can be written we obtain the following equations

$$ct' = ({A_0}^2 - p(\boldsymbol{a'})^2)ct - 2A_0 p < \boldsymbol{r}, \boldsymbol{a'} >$$

$$t' = ({A_0}^2 - p(\boldsymbol{a'})^2)t - 2A_0 \frac{p}{c} < \boldsymbol{r}, \boldsymbol{a'} > \tag{4.12}$$

and

$$I\boldsymbol{r'} = I\boldsymbol{r}({A_0}^2 + p(\boldsymbol{a'})^2) + 2IctA_0\boldsymbol{a'} - 2Ip\boldsymbol{a'} < \boldsymbol{a'}, \boldsymbol{r} >$$

$$\boldsymbol{r'} = \boldsymbol{r}({A_0}^2 + p(\boldsymbol{a'})^2) + 2ctA_0\boldsymbol{a'} - 2p\boldsymbol{a'} < \boldsymbol{a'}, \boldsymbol{r} > . \tag{4.13}$$

For the elliptic biquaternion $\mathbf{Q}$ the definitions given in the equations 4.3-4.4 using in the equation 4.12 we obtain

$$t' = (\cosh^2(p\frac{\theta_p}{2}) + \sinh^2(p\frac{\theta_p}{2})t - 2\cosh(p\frac{\theta_p}{2})\frac{p}{c}\frac{1}{p} < \boldsymbol{r}, \hat{q} > \sinh(p\frac{\theta_p}{2}). \tag{4.14}$$

Furthermore, using the equations of 4.6 and the known hyperbolic trigonometric function we obtain the expression 4.14 as follows:

$$t' = t\cosh(p\theta_p) - \frac{I}{c}\sinh(p\theta_p)\frac{<\boldsymbol{r},\vec{v}>}{\sqrt{<\vec{v},\vec{v}>}}.$$

Here

$$\hat{q} = I\frac{\vec{v}}{\sqrt{<\vec{v},\vec{v}>}} \tag{4.15}$$

therefore we obtain

$$t' = t\beta - I\frac{\beta}{c}\frac{|\nu|}{c}\frac{<\boldsymbol{r},\vec{v}>}{\sqrt{<\vec{v},\vec{v}>}} = \beta(t - \frac{I}{c^2} < \boldsymbol{r}, \vec{v} >). \tag{4.16}$$

Similarly, for the equality $\boldsymbol{r'}$ given in 4.13 we get

$$\begin{aligned}\boldsymbol{r'} &= \boldsymbol{r}(\cosh^2(p\frac{\theta_p}{2}) + p < \hat{q}, \hat{q} > \sinh^2(p\frac{\theta_p}{2}) + 2ct\cosh(p\frac{\theta_p}{2})\frac{\hat{q}}{p}\sinh(p\frac{\theta_p}{2}) \\ &-2p\frac{\hat{q}}{p}\sinh(p\frac{\theta_p}{2}) < \frac{\hat{q}}{p}\sinh(p\frac{\theta_p}{2}), \boldsymbol{r} > .\end{aligned} \tag{4.17}$$

Writing the equation 4.17 in the last equation we find

$$\boldsymbol{r'} = \boldsymbol{r} + ct\frac{I}{p}\frac{\vec{v}}{\sqrt{<\vec{v},\vec{v}>}}2\cosh(p\frac{\theta_p}{2})\sinh(p\frac{\theta_p}{2}) - 2\frac{1}{p}\sinh^2(p\frac{\theta_p}{2})I^2\frac{<\vec{v},\boldsymbol{r}>}{\sqrt{<\vec{v},\vec{v}>}\sqrt{<\vec{v},\vec{v}>}}$$

$$-2\sinh^2(p\frac{\theta_p}{2})\frac{I^2}{p}\frac{1}{\sqrt{<v,v>}\sqrt{<v,v>}}\vec{v} < \vec{v}, \boldsymbol{r} >$$

with the help of trigonometric identities we obtain

$$\boldsymbol{r'} = \boldsymbol{r} + \frac{1}{I}\frac{1}{\sqrt{1-p\nu^2}}\vec{v}t - \frac{1}{\sqrt{1-p\nu^2}}\frac{\vec{v}<\vec{v},\boldsymbol{r}>}{\nu^2} + \frac{\vec{v}<\vec{v},\boldsymbol{r}>}{\nu^2}.$$

Thus

$$\boldsymbol{r'} = x + \frac{1}{I}\frac{1}{\sqrt{1-|p|v^2}}\vec{v}t \pm \frac{x}{\sqrt{1-|p|v^2}} \mp x = \frac{1}{I}\frac{1}{\sqrt{1-|p|v^2}}\vec{v}t + \frac{x}{\sqrt{1-|p|v^2}}$$

is obtained. Consequently, the equation is obtained as follows:

$$\boldsymbol{r'} = \frac{I}{I^2}\frac{\vec{v}t}{\sqrt{1-|p|\,v^2}} + \frac{x}{\sqrt{1-|p|\,v^2}} = \frac{1}{\sqrt{1-|p|\,v^2}}(x - \frac{I}{|p|}\vec{v}t) = \beta(x - \frac{I}{|p|}\vec{v}t). \tag{4.18}$$

Here is $\boldsymbol{r} = +x$, $v = -x$, $< v, v >= \mp\nu^2$. These obtained equations for t' and $\boldsymbol{r'}$ are in conformity with the usual Lorentz transformation equations. It is also possible to give the matrix equality 4.9 which defines relativistic transformation carrying out with 4-dimensional matrix representations. For this purpose, if the elliptic biquaternion matrix definition in equality 3.4 is used, the equality in the form $\mathbf{R}' = \mathbf{Q}\mathbf{R}\bar{\mathbf{Q}}^*$ can be written. We give the matrix representation of this transformation using the matrix multiplication relations 3.5 and 3.9 since the matrices do not provide the commutative property. We get a unit elliptic biquaternion as $\mathbf{Q} = a_0 + I\boldsymbol{a'}$ $(a_1 = a_2 = a_3 = a'_0 = 0)$ and $I^2 = p < 0$ then

$$\mathbb{H}^-(\mathbf{R}') = \mathbb{H}^-(\mathbf{Q})\ \mathbf{R}\mathbb{H}^-(\bar{\mathbf{Q}}^*) = \mathbb{H}^-(\mathbf{Q})\mathbb{H}^+(\bar{\mathbf{Q}}^*)\ \mathbf{R} \tag{4.19}$$

$$\begin{bmatrix} R'_0 \\ \boldsymbol{R'} \end{bmatrix} = \begin{bmatrix} A_0 & -\boldsymbol{A} \\ \boldsymbol{A^T} & A_0 I_3 + \tilde{\boldsymbol{A}} \end{bmatrix} \begin{bmatrix} R_0 & 0 \\ 0 & R_0 I_3 \end{bmatrix} \begin{bmatrix} A_0 \\ \boldsymbol{A} \end{bmatrix}$$

is obtained. This equation can be written more clearly in type 4×4 as follows:

$$\begin{bmatrix} R'_0 \\ R'_1 \\ R'_2 \\ R'_3 \end{bmatrix} = \begin{bmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & A_3 & -A_2 \\ A_2 & -A_3 & A_0 & A_1 \\ A_3 & A_2 & -A_1 & A_0 \end{bmatrix} \begin{bmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & A_2 \\ A_2 & A_3 & A_0 & -A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix} \begin{bmatrix} R_0 \\ R_1 \\ R_2 \\ R_3 \end{bmatrix} \tag{4.20}$$

the equation

$$\mathbf{U} = \begin{bmatrix} {A_0}^2 - {A_1}^2 - {A_2}^2 - {A_3}^2 & -2A_0A_1 & -2A_0A_2 & -2A_0A_3 \\ 2A_0A_1 & {A_0}^2 - {A_1}^2 + {A_2}^2 + {A_3}^2 & -2A_1A_2 & -2A_1A_3 \\ 2A_0A_2 & -2A_1A_2 & {A_0}^2 + {A_1}^2 - {A_2}^2 + {A_3}^2 & -2A_2A_3 \\ 2A_0A_3 & -2A_1A_3 & -2A_2A_3 & {A_0}^2 + {A_1}^2 + {A_2}^2 - {A_3}^2 \end{bmatrix}$$

is concluded. Since the elliptic biquaternion $\mathbf{Q}$ is the unit elliptic biquaternion the above matrix relation can be written in a simpler form as

$$\begin{bmatrix} R'_0 \\ R'_1 \\ R'_2 \\ R'_3 \end{bmatrix} = \begin{bmatrix} 2{A_0}^2 - 1 & -2A_0A_1 & -2A_0A_2 & -2A_0A_3 \\ 2A_0A_1 & 1 - 2{A_1}^2 & -2A_1A_2 & -2A_1A_3 \\ 2A_0A_2 & -2A_1A_2 & 1 - 2{A_2}^2 & -2A_2A_3 \\ 2A_0A_3 & -2A_1A_3 & -2A_2A_3 & 1 - 2{A_3}^2 \end{bmatrix} \begin{bmatrix} R_0 \\ R_1 \\ R_2 \\ R_3 \end{bmatrix}$$

$$\begin{bmatrix} R'_0 \\ Ir'_1 \\ Ir'_2 \\ Ir'_3 \end{bmatrix} = \begin{bmatrix} 2{A_0}^2 - 1 & -2A_0A_1 & -2A_0A_2 & -2A_0A_3 \\ 2A_0A_1 & 1 - 2{A_1}^2 & -2A_1A_2 & -2A_1A_3 \\ 2A_0A_2 & -2A_1A_2 & 1 - 2{A_2}^2 & -2A_2A_3 \\ 2A_0A_3 & -2A_1A_3 & -2A_2A_3 & 1 - 2{A_3}^2 \end{bmatrix} \begin{bmatrix} R_0 \\ Ir_1 \\ Ir_2 \\ Ir_3 \end{bmatrix}.$$

In this equation we get

$$\mathbf{U}=\left[\begin{array}{cccc} 2{A_0}^2-1 & -2A_0A_1 & -2A_0A_2 & -2A_0A_3 \\ 2A_0A_1 & 1-2{A_1}^2 & -2A_1A_2 & -2A_1A_3 \\ 2A_0A_2 & -2A_1A_2 & 1-2{A_2}^2 & -2A_2A_3 \\ 2A_0A_3 & -2A_1A_3 & -2A_2A_3 & 1-2{A_3}^2 \end{array}\right].$$

If the matrix equality is defined as above then using definitions 4.6 is obtained as

$$\mathbf{U}=\left[\begin{array}{cccc} \beta & \pm\frac{\beta}{c}v_1 & \pm\frac{\beta}{c}v_2 & \pm\frac{\beta}{c}v_3 \\ \mp\frac{\beta}{c}v_1 & 1-(\beta-1)\frac{{v_1}^2}{\mp\nu^2} & -(\beta-1)\frac{v_1v_2}{\mp\nu^2} & -(\beta-1)\frac{v_1v_3}{\mp\nu^2} \\ \mp\frac{\beta}{c}v_2 & -(\beta-1)\frac{v_1v_2}{\pm\nu^2} & 1-(\beta-1)\frac{{v_2}^2}{\mp\nu^2} & -(\beta-1)\frac{v_2v_3}{\mp\nu^2} \\ \mp\frac{\beta}{c}v_3 & -(\beta-1)\frac{v_1v_3}{\pm\nu^2} & -(\beta-1)\frac{v_2v_3}{\mp\nu^2} & 1-(\beta-1)\frac{{v_3}^2}{\mp\nu^2} \end{array}\right].$$

Thus, it is possible to represent equality 4.19 as

$$\mathbb{H}^-(\mathbf{R}')=\mathbf{U}\ \mathbf{R}.$$

In addition, it is possible to indicate the equation defined in 4.19 with eight-dimensional real matrices.

We can write as $\mathbb{H}^-(\mathbf{R}')=\mathbb{H}^-(\mathbf{Q})\ \mathbf{R}\ \mathbb{H}^-(\bar{\mathbf{Q}}^*)=\mathbb{H}^-(\mathbf{Q})\mathbb{H}^-(\breve{\bar{\mathbf{Q}}}^{*})\ \mathbf{R}$. This equation can be written as

$$\mathbb{H}^-(\mathbf{R}')=\left[\begin{array}{cccc} a_0 & 0 & a'_0 & -a' \\ 0 & a_0I_3 & a'T & a'_0I_3-\tilde{a}' \\ -a'_0 & a' & a_0 & 0 \\ -a'^T & -a'_0I_3-\tilde{a}' & 0 & a_0I_3 \end{array}\right]\left[\begin{array}{cccc} a_0 & 0 & a'_0 & -a' \\ 0 & a_0I_3 & a'^T & {a'_0}^TI_3-\tilde{a}' \\ -a'_0 & a' & a_0 & 0 \\ -a'^T & -a'_0I_3+\tilde{a}' & 0 & a_0I_3 \end{array}\right]\left[\begin{array}{c} R_0 \\ 0 \\ 0 \\ R \end{array}\right].$$

Also, it can be obtained more clearly as

$$\left[\begin{array}{c} ct' \\ 0 \\ 0 \\ 0 \\ 0 \\ r'_1 \\ r'_2 \\ r'_3 \end{array}\right]=\left[\begin{array}{cccccccc} a_0 & 0 & 0 & 0 & a'_0 & -a'_1 & -a'_2 & -a'_3 \\ 0 & a_0 & 0 & 0 & a'_1 & a'_0 & a'_3 & -a'_2 \\ 0 & 0 & a_0 & 0 & a'_2 & -a'_3 & a'_0 & a'_1 \\ 0 & 0 & 0 & a_0 & a'_3 & a'_2 & -a'_1 & a'_0 \\ -a'_0 & a'_1 & a'_2 & a'_3 & a_0 & 0 & 0 & 0 \\ -a'_1 & a'_0 & -a'_3 & a'_2 & 0 & a_0 & 0 & 0 \\ -a'_2 & a'_3 & a'_0 & -a'_1 & 0 & 0 & a_0 & 0 \\ -a'_3 & -a'_2 & a'_1 & -a'_0 & 0 & 0 & 0 & a_0 \end{array}\right]\left[\begin{array}{cccccccc} a_0 & 0 & 0 & 0 & a'_0 & -a'_1 & -a'_2 & -a'_3 \\ 0 & a_0 & 0 & 0 & a'_1 & a'_0 & -a'_3 & a'_2 \\ 0 & 0 & a_0 & 0 & a'_2 & a'_{33} & a'_0 & -a'_1 \\ 0 & 0 & 0 & a_0 & a'_3 & -a'_2 & a'_1 & a'_0 \\ -a'_0 & a'_1 & a'_2 & a'_3 & a_0 & 0 & 0 & 0 \\ -a'_1 & -a'_0 & a'_3 & -a'_2 & 0 & a_0 & 0 & 0 \\ -a'_2 & -a'_3 & -a'_0 & a'_1 & 0 & 0 & a_0 & 0 \\ -a'_3 & a'_2 & -a'_1 & -a'_0 & 0 & 0 & 0 & a_0 \end{array}\right]\left[\begin{array}{c} ct \\ 0 \\ 0 \\ 0 \\ 0 \\ r_1 \\ r_2 \\ r_3 \end{array}\right].$$

From here two matrix multiplications

$$\mathbb{H}^-(\mathbf{Q})\,\mathbb{H}^-\left(\breve{\mathbf{Q}}\right)\cong$$

$$\left[\begin{array}{cccccccc} {a_0}^2+{a'_1}^2+{a'_2}^2+{a'_3}^2 & 0 & 0 & 0 & 0 & -2a_0a'_1 & -2a_0a'_2 & -2a_0a'_3 \\ 0 & {a_0}^2+{a'_1}^2-{a'_2}^2-{a'_3}^2 & 2a'_1a'_2 & 2a'_1a'_3 & 2a_0a'_1 & 0 & 0 & 0 \\ 0 & 2a'_1a'_2 & {a_0}^2-{a'_1}^2+{a'_2}^2-{a'_3}^2 & 2a'_2a'_3 & 2a_0a'_2 & 0 & 0 & 0 \\ 0 & 2a'_1a'_3 & 2a'_2a'_3 & {a_0}^2-{a'_1}^2-{a'_2}^2+{a'_3}^2 & 2a_0a'_3 & 0 & 0 & 0 \\ 0 & 2a_0a'_1 & 2a_0a'_2 & 2a_0a'_3 & {a_0}^2+{a'_1}^2+{a'_2}^2+{a'_3}^2 & 0 & 0 & 0 \\ -2a'_1a_0 & 0 & 0 & 0 & 0 & {a_0}^2+{a'_1}^2-{a'_2}^2-{a'_3}^2 & 2a'_1a'_2 & 2a'_1a'_3 \\ -2a'_2a_0 & 0 & 0 & 0 & 0 & 2a'_1a'_2 & {a_0}^2-{a'_1}^2+{a'_2}^2-{a'_3}^2 & 2a'_2a'_3 \\ -2a'_3a_0 & 0 & 0 & 0 & 0 & 2a'_1a'_3 & 2a'_2a'_3 & {a_0}^2-{a'_1}^2-{a'_2}^2+{a'_3}^2 \end{array}\right]$$

is obtained. We can give this equation more clearly as follows:

$$\mathbb{H}^-(\mathbf{Q})\,\mathbb{H}^-\left(\breve{\mathbf{Q}}\right)\cong$$

$$
\begin{bmatrix}
\frac{(p-1){a_0}^2+1}{p} & 0 & 0 & 0 & 0 & -2a_0{a'}_1 & -2a_0{a'}_2 & -2a_0{a'}_3 \\
0 & \frac{{a_0}^2(p+1)+2p{a'}^2{}_1-1}{p} & 2{a'}_1{a'}_2 & 2{a'}_1{a'}_3 & 2a_0{a'}_1 & 0 & 0 & 0 \\
0 & 2{a'}_1{a'}_2 & \frac{{a_0}^2(p+1)+2p{a'}_2{}^2-1}{p} & 2{a'}_2{a'}_3 & 2a_0{a'}_2 & 0 & 0 & 0 \\
0 & 2{a'}_1{a'}_3 & 2{a'}_2{a'}_3 & \frac{{a_0}^2(p+1)+2p{a'}_3{}^2-1}{p} & 2a_0{a'}_3 & 0 & 0 & 0 \\
0 & 2a_0{a'}_1 & 2a_0{a'}_2 & 2{a'}_3a_0 & \frac{(p-1){a_0}^2+1}{p} & 0 & 0 & 0 \\
-2a_0{a'}_1 & 0 & 0 & 0 & 0 & \frac{{a_0}^2(p+1)+2p{a'}^2{}_1-1}{p} & 2{a'}_1{a'}_2 & 2{a'}_1{a'}_3 \\
-2a_0{a'}_2 & 0 & 0 & 0 & 0 & 2{a'}_1{a'}_2 & \frac{{a_0}^2(p+1)+2p{a'}_2{}^2-1}{p} & 2{a'}_2{a'}_3 \\
-2a_0{a'}_3 & 0 & 0 & 0 & 0 & 2{a'}_1{a'}_3 & 2{a'}_2{a'}_3 & \frac{{a_0}^2(p+1)+2p{a'}_3{}^2-1}{p}
\end{bmatrix}. \tag{4.21}
$$

For the real components of the elliptic biquaternion as can be seen from 4.6;

$$
a_0 = \cosh(\frac{I\theta_p}{2}) \ , \ {a_1}' = \frac{1}{I}\frac{\vec{v}_1}{\nu}\sinh(\frac{I\theta_p}{2}) \ \ , \ {a_2}' = \frac{1}{I}\frac{\vec{v}_2}{\nu}\sinh(\frac{I\theta_p}{2}) \ , \ a_3' = \frac{1}{I}\frac{\vec{v}_3}{\nu}\sinh(\frac{I\theta_p}{2})
$$

and the transformation matrix $\mathbf{U}$ that provides definitions given in 4.6 is

$$
\mathbf{U} \cong \begin{bmatrix}
\frac{(\beta+1)(p-1)+2}{2p} & 0 & 0 & 0 & 0 & \pm\frac{1}{I}\frac{\beta}{c}v_1 & \pm\frac{1}{I}\frac{\beta}{c}v_2 & \pm\frac{1}{I}\frac{\beta}{c}v_3 \\
0 & \frac{(\beta+1)(p+1)+2(\beta-1)\frac{{v_1}^2}{\mp\nu^2}-2}{2p} & \frac{(\beta-1)v_1v_2}{\mp p\nu^2} & \frac{-(\beta-1)v_1v_3}{\mp p\nu^2} & \mp\frac{1}{I}\frac{\beta}{c}v_1 & 0 & 0 & 0 \\
0 & \frac{(\beta-1)v_1v_2}{\mp pv^2} & \frac{(\beta+1)(p+1)+2(\beta-1)\frac{{v_2}^2}{\mp\nu^2}-2}{2p} & \frac{(\beta-1)v_2v_3}{\mp p\nu^2} & \mp\frac{1}{I}\frac{\beta}{c}v_2 & 0 & 0 & 0 \\
0 & \frac{(\beta-1)v_1v_3}{\mp pv^2} & \frac{(\beta-1)+v_2v_3}{\mp pv^2} & \frac{(\beta+1)(p+1)+2(\beta-1)\frac{{v_3}^2}{\mp\nu^2}-2}{2p} & \mp\frac{1}{I}\frac{\beta}{c}v_3 & 0 & 0 & 0 \\
0 & \mp\frac{1}{I}\frac{\beta}{c}v_1 & \mp\frac{1}{I}\frac{\beta}{c}v_2 & \mp\frac{1}{I}\frac{\beta}{c}v_3 & \frac{(p-1)(\beta+1)+2}{2p} & 0 & 0 & 0 \\
\pm\frac{1}{I}\frac{\beta}{c}v_1 & 0 & 0 & 0 & 0 & \frac{(\beta+1)(p+1)+2(\beta-1)\frac{{v_1}^2}{\mp\nu^2}-2}{2p} & \frac{(\beta-1)v_1v_2}{\mp pv^2} & \frac{(\beta-1)v_1v_3}{\mp pv^2} \\
\pm\frac{1}{I}\frac{\beta}{c}v_2 & 0 & 0 & 0 & 0 & \frac{(\beta-1)v_1v_2}{\mp p\nu^2} & \frac{(\beta+1)(p+1)+2(\beta-1)\frac{{v_2}^2}{\nu^2}-2}{2p} & \frac{(\beta-1)v_2v_3}{\mp pv^2} \\
\pm\frac{1}{I}\frac{\beta}{c}v_3 & 0 & 0 & 0 & 0 & \frac{(\beta-1)v_1v_3}{\mp pv^2} & \frac{(\beta-1)v_2v_3}{\mp p\nu^2} & \frac{(\beta+1)(p+1)+2(\beta-1)\frac{{v_3}^2}{\mp\nu^2}-2}{2p}
\end{bmatrix}
$$

and

$$
\mathbb{H}^-(\mathbf{R}') \cong \begin{bmatrix}
\left(\frac{(\beta+1)(p-1)-2}{2p}\right)ct \pm \frac{1}{I}\beta v_1r_1 \pm \frac{1}{I}\beta v_2r_2 \pm \frac{1}{I}\beta v_3r_3 \\
0 \\
0 \\
0 \\
0 \\
\pm\frac{1}{I}\beta ctv_1 + \left(\frac{(\beta+1)(p+1)+2(\beta-1)\frac{{v_1}^2}{\mp\nu^2}-2}{2p}\right)r_1 + \left(\frac{(\beta-1)v_1v_2}{\mp pv^2}\right)r_2 + \left(\frac{(\beta-1)v_1v_3}{\mp pv^2}\right)r_3 \\
\pm\frac{1}{I}\beta ctv_1 + \left(\frac{(\beta-1)v_1v_2}{\mp pv^2}\right)r_1 + \left(\frac{(\beta+1)(p+1)+2(\beta-1)\frac{{v_2}^2}{\mp\nu^2}-2}{2p}\right)r_2 + \left(\frac{(\beta-1)v_2v_3}{\mp pv^2}\right)r_3 \\
\pm\frac{1}{I}\beta ctv_3 + \left(\frac{(\beta-1)v_1v_3}{\mp pv^2}\right)r_1 + \left(\frac{(\beta-1)v_2v_3}{\mp pv^2}\right)r_2 + \left(\frac{(\beta+1)(p+1)+2(\beta-1)\frac{{v_3}^2}{\mp\nu^2}-2}{2p}\right)r_3
\end{bmatrix}
$$

is obtained.

Hence, the algebraic equation of Lorentz transformations is obtained from the desired matrix representation. Also in the transformation matrix in above for $p = I^2 = -1$, $c = 1$ we obtain following as

$$
\mathbf{U} = \begin{bmatrix}
\beta t & 0 & 0 & 0 & 0 & \mp i\beta v_1 & \mp i\beta v_2 & \mp i\beta_3 v_3 \\
0 & \frac{-(\beta-1){v_1}^2}{\mp\nu^2}+1 & \frac{-(\beta-1)v_1v_2}{\mp\nu^2} & \frac{-(\beta-1)v_1v_3}{\mp\nu^2} & \mp i\beta v_1 & 0 & 0 & 0 \\
0 & \frac{-(\beta-1)v_1v_2}{\mp\nu^2} & \frac{-(\beta-1){v_2}^2}{\mp\nu^2}+1 & \frac{-(\beta-1){v_2}^2}{\mp\nu^2} & \mp i\beta v_2 & 0 & 0 & 0 \\
0 & \frac{-(\beta-1)v_1v_3}{\mp\nu^2} & \frac{-(\beta-1)v_2v_3}{\mp\nu^2} & \frac{-(\beta-1){v_3}^2}{\mp\nu^2}+1 & \mp i\beta v_3 & 0 & 0 & 0 \\
0 & \pm i\beta v_1 & \pm i\beta v_2 & \pm i\beta v_3 & \beta & 0 & 0 & 0 \\
\mp i\beta v_1 & 0 & 0 & 0 & 0 & \frac{-(\beta-1){v_1}^2}{\mp\nu^2}+1 & \frac{-(\beta-1)v_1v_2}{\mp\nu^2} & \frac{-(\beta-1)v_1v_3}{\mp\nu^2} \\
\mp i\beta v_2 & 0 & 0 & 0 & 0 & \frac{-(\beta-1)v_1v_2}{\mp\nu^2} & \frac{-(\beta-1){v_2}^2}{\mp\nu^2}+1 & \frac{-(\beta-1)v_2v_3}{\nu^2} \\
\mp i\beta v_3 & 0 & 0 & 0 & 0 & \frac{-(\beta-1)v_1v_3}{\mp\nu^2} & \frac{-(\beta-1)v_2v_3}{\mp\nu^2} & \frac{-(\beta-1){v_3}^2}{\mp\nu^2}+1
\end{bmatrix}.
$$

Also, it is obtained by taking $<\vec{v},\vec{v}>=-\nu^2$ by and a velocity $\vec{v}=v_1e_1+v_2e_2+v_3e_3$ in the reverse direction generally as follows:

$$\mathbf{U}=\begin{bmatrix} \beta & 0 & 0 & 0 & 0 & -i\beta v_1 & -i\beta v_2 & -i\beta_3 v_3 \\ 0 & \frac{(\beta-1){v_1}^2}{\nu^2}+1 & \frac{(\beta-1)v_1v_2}{\nu^2} & \frac{(\beta-1)v_1v_3}{\nu^2} & i\beta v_1 & 0 & 0 & 0 \\ 0 & \frac{(\beta-1)v_1v_2}{\nu^2} & \frac{(\beta-1){v_2}^2}{\nu^2}+1 & \frac{(\beta-1){v_2}^2}{\nu^2} & i\beta v_2 & 0 & 0 & 0 \\ 0 & \frac{(\beta-1)v_1v_3}{\nu^2} & \frac{(\beta-1)v_2v_3}{\nu^2} & \frac{(\beta-1){v_3}^2}{\nu^2}+1 & i\beta v_3 & 0 & 0 & 0 \\ 0 & i\beta v_1 & i\beta v_2 & i\beta v_3 & \beta & 0 & 0 & 0 \\ -i\beta v_1 & 0 & 0 & 0 & 0 & \frac{(\beta-1){v_1}^2}{\nu^2}+1 & \frac{(\beta-1)v_1v_2}{\nu^2} & \frac{(\beta-1)v_1v_3}{\nu^2} \\ -i\beta v_2 & 0 & 0 & 0 & 0 & \frac{(\beta-1)v_1v_2}{\nu^2} & \frac{-(\beta-1){v_2}^2}{\mp\nu^2}+1 & \frac{(\beta-1)v_2v_3}{\nu^2} \\ -i\beta v_3 & 0 & 0 & 0 & 0 & \frac{(\beta-1)v_1v_3}{\nu^2} & \frac{(\beta-1)v_2v_3}{\nu^2} & \frac{(\beta-1){v_3}^2}{\nu^2}+1 \end{bmatrix} [20].$$

5 Conclusions

In this study, the well-known Lorentz transformations in the physics were studied with elliptic biquaternions. Transformation matrices of 4×4 type consisting of elliptic components and 8×8 type consisting of real components were defined with the equality of the elliptic biquaternion $\mathbf{R}=ct+I\boldsymbol{r}$ combining space and time. The matrix equality 4.8 defining the relativistic transformation of $4-$dimensional matrix expressions was given in 4.20. We also gave an easier and rather useful the expression of transformation $\mathbf{U}$ through the elliptic biquaternionic transformation matrix that we defined. Thanks to by expressing its matrix containing elliptic components in 8×8 type, we executed that mathematical expressions are obtained with fewer operations by making it possible to express the transformation of the relativistic transformation relation in the 8×8 type matrix representation in a simpler way. The recommended method may be preferred according to the type of operation, depending on the reader preference in order to be clear and simple. The form of expression in 4.1 and the physical results obtained with the help of 4.16 and 4.18 are provided easily interrelated. It is possible to express basic equations of electromagnetism with many physical quantities of different properties such as electric and magnetic fields, energy and momentum quantities, electrical current and charge density, scalar and vector potential which are closely related to each other through the equations and matrices obtained through this study, by means of elliptic biquaternion. Therefore, the expressions obtained are important in terms of being versatile and obtaining relativistic equations which very important in physics.

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